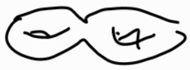
Fundamental groups of low-coregularity pairs.

joint w/ Lukas Braun

Dimension 1 | Curves. / $\mathbb{C}$. (smooth).

genus		$\pi_1(X)$	
0		$\{1\}$	Trivial
1		$\mathbb{Z}\times\mathbb{Z}$.	Abelian
≥ 2		$\langle \alpha_1, \beta_1 \dots \alpha_g, \beta_g \mid [\alpha_1 \beta_1] \dots [\alpha_g \beta_g] \rangle$	non-abelian.

Higher dimensions | Smooth.

Fano	K_X anti-ample	$\pi_1(X)$ is trivial.
Calabi-Yau	$K_X \equiv 0$	$\pi_1(X)$ is virtually abelian $A \triangleleft \pi_1(X)$, A is abelian $\lvert \pi_1(X) : A \rvert < \infty$
Canonically Polarized	K_X ample	More complicated.

What happens when we add singularities?

Singularities of the MMP.

- log canonical.
- klt.

The local topology of the singularities matters!!

Theorem (Braun 2021)

Let $(X;x)$ be a klt singularity. Then $\pi_1^{loc}(X;x) = \pi_1(B(x)\setminus\{x\})$ is always finite.

Theorem (Braun 2021)

Let X be a Fano klt variety. Then the $\pi_1(X_{reg})$ is finite.

klt Calabi-Yau:

Theorem (Campana-Claudon 2014)

Let X be a klt Calabi-Yau surface, then $\pi_1(X_{reg})$ is virtually abelian.

Pairs $(X, \underbrace{D}_{\mathbb{Q}})$.

Standard approximation:

$$D_{st} := \max \left\{ \sum \underbrace{\left(1 - \frac{1}{m_i}\right)}_{} D_i \leq D \right\}.$$

$m_i \in \mathbb{Z}^+$
or $\left(1 - \frac{1}{m_i}\right) = 1$.

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§ The orbifold fundamental group of the smooth locus of a pair.

$$\pi_1^{reg}(X, D) = \pi_1\left(X_{reg} \setminus \mathrm{Sup}(D_{st})\right) / N$$

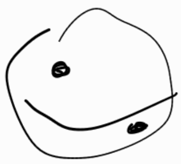
N is the normal group generated by loops $(\gamma_i^{m_i})$. (Where γ_i is a loop around D_i with $\mathrm{coeff}_{D_i}(D_{st}) = 1 - \frac{1}{m_i}$. i.e. if coeff is 1, $(\gamma_i^{m_i})$ does not appear on N.

Example: $(C,0)$ $\quad C$ ell. curve

$$\pi_1^{reg}(C,0) = \pi_1(C) = \mathbb{Z}\times\mathbb{Z}.$$

$\pi_1^{reg}(\mathbb{P}^1, \{0\}+\{\infty\})$ is log canonical C-Y.

$$= \pi_1(\mathbb{P}^1\setminus\{0,\infty\}) = \mathbb{Z} \quad \text{abelian.}$$



- $\pi_1^{reg}(\mathbb{P}^1, (1-\frac{1}{\alpha})\{0\}+(1-\frac{1}{\alpha})\{\infty\})$ ← klt Fano.

$$\pi_1(\mathbb{P}^1\setminus\{0,\infty\})/\langle\gamma_1^{\alpha},\gamma_2^{\alpha}\rangle$$

$$= \mathbb{Z}/\alpha\mathbb{Z}. \quad \text{finite.}$$

Theorem (Braun 2021)

Let (X,D) be a klt Fano pair, then $\pi_1^{reg}(X,D)$ is finite.

Theorem (Campana - Claudon 2014)

Let (X, D) be a C-Y. surface pair with standard coeff. Then

$\pi_1^{reg}(X, D)$ is virtually abelian

with rank at most 4.

Coregularity) Calabi-Yau type pairs.

A pair (X,D) is of Calabi-Yau type, if $\exists B \geq D$ s.t. (X,B) is (log canonical) Calabi-Yau.

Dual complex of (X,D).

{all of our pairs are l.c.}

- Let $(Y,D_Y) \to (X,D)$ be a resolution.

$D_Y^{=1} := E_1 + \dots + E_r$.

$\mathcal{D}(Y,D_Y) :=$

- For E_i, $\to$ there exists a vertex S_{E_i}
- For $I \subseteq \{1,\dots,r\}$ and irr. component $\bigcap_I E_i$ $\to$ a $(|I|-1)$-dim. simplex S_Z.
- glue by inclusions...

$\mathcal{D}(X,D)$ is the homotopy class of $\mathcal{D}(Y,D_Y)$.

For (X,B) a C-Y. pair.

{$\dim \emptyset = -1$}

$$\operatorname{coreg}(X,B) := (\dim X - 1) - \dim \mathcal{D}(X,B)$$

$$0 \leq \operatorname{coreg}(X,B) \leq \dim X.$$

klt case.

- For (X,D) pair $\{\text{C-Y type}\}$

$$\widehat{\operatorname{coreg}}(X,D) := \inf \left(\operatorname{coreg}(X,B) \;\middle|\; \begin{array}{l} B \geq D \\ (X,B) \text{ lc. C-Y.} \end{array} \right)$$

$0 \leq \widehat{\operatorname{coreg}} \leq \dim X$, and $\widehat{\operatorname{coreg}}(X,B) = \infty$ if not C-Y type.

Main Theorems

Theorem 1 Braun–F. 2024

Let (X,D) be a klt pair with $\widehat{\mathrm{coreg}}(X,D)=0$.
Then $\pi_1^{reg}(X,D)$ is finite.

Theorem 2 Braun. F. 2024

Let $c\in\{1,2\}$.
Let (X,D) be a klt pair with $\widehat{\mathrm{coreg}}(X,D)=c$.
Then $\pi_1^{reg}(X,D)$ is virtually abelian of rank at most $2c$.

Proof of Theorems

☆ They hold if we put $\dim = c$, and standard <u>coefficients</u>.

☆ We reduce to the case of std <u>coeff</u>.

If Thm 1-2 hold for std. $\Rightarrow$

Let (X, D) with $\widehat{\mathrm{coreg}}(X, D) = c$.

$\Rightarrow (X, D_{st})$, $D_{st} \leq D$,

$\widehat{\mathrm{coreg}}(X, D_{st}) \leq \widehat{\mathrm{coreg}}(X, D) = c$.

& $\pi_1^{reg}(X, D) = \pi_1^{reg}(X, D_{st})$, the Thm holds (std. coeff.).

$\boxed{1^{st} \text{ Case}}$ If (X, D) is Calabi-Yau.

- $\mathrm{coreg}(X, D) = c$ and (X, D) is klt
  $\Rightarrow \dim X = c \in \{0, 1, 2\}$.

D with std. coeff. $\Rightarrow$ The result from Campana-Claudon for surfaces or the case of curves.

$\boxed{2^{nd} \text{ Case}}$ (X, D) is not Calabi-Yau.

$\exists\, B > D$ $\{B \geq D,\ B \neq D\}$ s.t.

(X, B) is C-Y with $\mathrm{coreg}(X, B) = c$.

((X, B) is log canonical)

★ Take a $\mathbb{Q}$-factorial dlt modification of (X,B).

$$\phi : (Y, B_Y) \to (X, B).$$

There exists some $\underbrace{D_Y} \leq P_Y < B_Y$.

s.t. $\pi_1^{reg}(Y, P_Y) \twoheadrightarrow \underbrace{\pi_1^{reg}(X, D)}$

with (Y, P_Y) klt.

$$\cancel{D_Y} \leq \underbrace{P_Y} \leq B_Y$$

(Y, P_Y) klt $\quad (Y, B_Y)$ dlt

$$K_Y + B_Y \sim_{\mathbb{Q}} 0.$$

$\underbrace{K_Y + P_Y} < K_Y + B_Y \sim_{\mathbb{Q}} 0$

is not pseff.

We can run $(K_Y + P_Y)$-MMP.

$$\begin{array}{ccc} Y & \dashrightarrow & Z \\ & & \downarrow{\scriptstyle f} \\ & & W \end{array}$$

$$\pi_1^{reg}(Z, P_Z) \twoheadrightarrow \pi_1^{reg}(Y, P_Y).$$

$$\boxed{\mathrm{coreg}(Z, B_Z) = \mathrm{coreg}(Y, B_Y) = \mathrm{coreg}(X, B)}$$

By the canonical bundle Formula:

We obtain (W, B_W) s.t.

$$f^*(K_W + B_W) \sim_{\mathbb{Q}} K_Z + B_Z.$$

We obtain:

$$\begin{array}{ccc} F & \to & Z \\ & & \downarrow \\ & & W \end{array}$$

Fano type Fibers.

$$\underbrace{\pi_1^{reg}(F, P_F)}_{\text{Fano}} \to \underbrace{\pi_1^{reg}(Z, P_Z)}_{\text{coreg} \leq c} \to \underbrace{\pi_1^{reg}(W, P_W)}_{\substack{\text{coreg} \leq c \\ \dim < \dim X}} \to 1$$

(Finite)

virt. abelian (by induction) of rank $\leq 2c$

$\Longrightarrow$ $\pi_1^{reg}(Z, P_Z)$ is virt. ab. of rank $\leq 2c$.

$$\pi_1^{reg}(Z, P_Z) \twoheadrightarrow \pi_1^{reg}(X, P_X).$$

$\square$

Effective

Theorem Braun–Filipazzi–Moraga–Svaldi 2022

Let (X, D) be a klt Fano pair of $\dim = d$.
$\pi_1^{reg}(X, D)$ is virt. abelian of rank $\leq d$ with index bounded $c(d)$.

Theorem Braun F. 2024

Let (X, D) klt pair with $\operatorname{coreg}(X, D) = c \leq 2$. and $\dim X = d$.
Then $\pi_1^{reg}(X, D)$ is virt. solvable of length $\leq 2d-1$, and index is bounded by $\boxed{i(c,d)}$.
